(Readers may remember that the above expression means $\mu_B(y_1) = 0.8$, $\mu_B(y_2) = 0.9$, and so on.) It is natural to take

$$\mu_{f^{-1}(B)}(x_1) = \mu_{f^{-1}(B)}(x_2) = 0.8$$

but for y_1, two values 0.1 and 0.5 are mapped from x_1 and x_2, respectively. Extension principle states that the maximum of the possible grades for x is defined to be the grade of the corrresponding y. Thus,

$$\mu_{f(A)}(y_1) = \max\{\mu_A(x_1), \mu_A(x_2)\} = 0.5$$

Generally, let A be a fuzzy set of X, then the image $f(A)$ is defined by

$$\mu_{f(A)}(y) = \begin{cases} \max_{y \in f-1(y)} \mu_A(x) & (f-1(y) \neq \emptyset) \\ 0 & (f-1(y) = \emptyset) \end{cases}$$

Extension principle in many textbooks are given in a more general form. Namely, let universal sets be $X_1, X_2, \ldots, X_n$ and fuzzy sets be $A_1, A_2, \ldots, A_n$ in the respective universes. Assume $f\colon X_1 \times X_2 \times \cdots \times X_n \to Y$ Then,

$$\begin{aligned} & \mu_{f(A_1 \times A_2 \times \cdots \times A_n)}(y) \\ = & \max_{y=f(x_1,x_2,\ldots,x_n)} \mu_{A_1 \times A_2 \times \cdots \times A_n}(x_1, x_2, \ldots, x_n) \\ = & \max_{y=f(x_1,x_2,\ldots,x_n)} \min\{\mu_{A_1}, \mu_{A_2}, \ldots, \mu_{A_n}\} \end{aligned}$$

Remark: In general, max in the right hand sides are replaced by sup in advanced textbooks. We assume here, however, that sup can always be replaced by max for simplicity, as noted earlier. This assumption for the simplification is satisfied in most of practical or applicational examples.

3.4 Fuzzy relations

Crisp or ordinary relations inlude equality (=), inequality(>, <), equaivalence relations for generating classes, and so on. As fuzzy sets are generalizations of crisp sets, fuzzy relations generalize crisp relations.

A relation (or more precisely, binary relation) R defined on X means that for any pairs of elements $x, y \in X$, one and only one of xRy (x and y have the relation R) or $x\bar{R}y$ (x and y do not have the relation R) holds. This form of the infix notation can be transformed into an prefix notation $R(x, y)$ of the same meaning. Thus, using R as a 0/1 valued function:

$$\begin{array}{lcl} xRy & \Longleftrightarrow & R(x,y) = 1 \\ x\bar{R}y & \Longleftrightarrow & R(x,y) = 0 \end{array}$$

For example, assume that $R_=(x, y)$ and $R_>(x, y)$ are the functions corresponding to equality $=$ and inequality $>$, respectively,

$$\begin{array}{lcl} R_=(x,y) = 1 & \Longleftrightarrow & x = y \\ R_=(x,y) = 0 & \Longleftrightarrow & x \neq y \\ R_>(x,y) = 1 & \Longleftrightarrow & x > y \\ R_=(x,y) = 0 & \Longleftrightarrow & x \leq y \end{array}$$

The latter form $R(x, y)$ of the binary relation is a 0/1 valued function, and furthermore, $R(x, y)$ is the characteristic function of the set of elements (x, y) such that xRy :

$$S_R = \{(x,y) \ : \ R(x,y) = 1\} = \{(x,y) \ : \ xRy\}$$

In this way, a relation R on X is equivalent to the set S_R in the product space $X \times X$. Figure 3.15 shows S_R in case when R means equality $=$.

Fuzzy relations have been considered for representing relations of "approximately equal" ($\approx$) or "far greater than" ($\gg$), and so on. Since a crisp relation is equivalent to a set in $X \times X$, a fuzzy relation should be defined as a fuzzy set in $X \times X$. Equivalently, $R(x, y)$ as a characteristic function of S_R should be generalized to a membership function defined on $X \times X$.

Thus, we define a fuzzy relation R on X to be a fuzzy set of the same symbol R of $X \times X$. The membership function for R is denoted by $\mu_R(x, y)$ or more simply $R(x, y)$ for any $x, y \in X$.

More generally, given universes X_1, X_2,. . . ,X_n, n-ary fuzzy relation R is defined to be a fuzzy set R of $X_1 \times X_2 \times \cdots \times X_n$. Its membership function is denoted by $\mu_R(x_1, x_2, \ldots, x_n)$ or $R(x_1, x_2, \ldots, x_n)$.

Let us consider a fuzzy relation $R_\approx$ for "approximately equal" relation. When $x = y$, $R_\approx(x, y) = 1$. Accordign as $|x - y|$ becomes greater, $R_\approx(x, y)$ decreases. There are many functions to satisfy these conditions, one of which is $\exp(-\frac{|x-y|}{C})$. Thus we can take

$$R_\approx(x,y) = \exp(-\frac{|x-y|}{C})$$

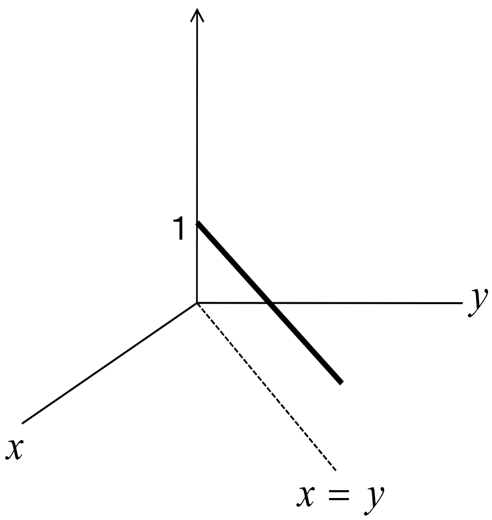

Figure 3.15: Characteristic function $R_{=}$ for equality relation

where C is an appropriate positive constant.

Figure 3.16 shows $R_{\approx}(x, y)$. It is clear that the graph of $R_{\approx}$ is symmetrical with respect to the plane $x = y$.

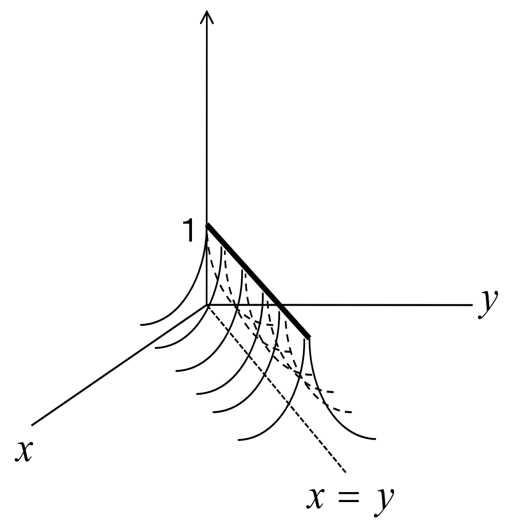

Figure 3.16: Fuzzy relation $R_{\approx}$

Well-known crisp relations are thus extended by specifying appropriate membership functions. Another important class of fuzzy relations are expressed using fuzzy graphs, which visualize fuzzy relations on finite sets.

Consider a finite universe $X = \{John, Mary, Thomas, Philip\}$ which is simplified as $X = \{J, M, T, P\}$ hereafter. In a previous section we have considered a similar universe and fuzzy set of that universe. Let us try to quantize "strength of friendship" between a pair of individuals in this universe. The most strong friendship is coded by 1.0. On the other hand, the weakest friendship is given the value 0.0. Other strength of friendship are given membership between these extreme values. Assume that the data are given by the following matrix.

$$R = \begin{array}{c} \\ J \\ M \\ T \\ P \end{array} \begin{array}{c} \begin{array}{cccc} J & M & T & P \end{array} \\ \begin{pmatrix} 1 & 0.5 & 0.9 & 0 \\ 0.5 & 1 & 0.4 & 0.3 \\ 0.9 & 0.4 & 1 & 0.7 \\ 0 & 0.3 & 0.7 & 1 \end{pmatrix} \end{array}$$

Friendship between an individual and himself is set to unity and the above matrix is symmetric. Now, assume that this matrix is a fuzzy relation of the same symbol: $R(John, John) = 1$, $R(Mary, Thomas) = R(Thomas, Mary) = 0.4$, and so on. As we define later, if the matrix of a fuzzy relation is symmetric and has unity as its diagonal elements, the relation is called symmetric and reflexive. Such a fuzzy relation can be expressed as an undirected graph with weights on the edges (Fig. 3.17).

Notice that an undirected graph is simply called a graph. When we regard such a weighted graph as a visualization of a fuzzy relation, the graph is called a fuzzy graph. Note that an edge with the grade 0 can be eliminated from the graph.

Let us consider another fuzzy relation of "much older than" on the same universe. Assume that this relation Q is given by the folowing matrix.

$$Q = \begin{array}{c} \\ J \\ M \\ T \\ P \end{array} \begin{array}{c} \begin{array}{cccc} J & M & T & P \end{array} \\ \begin{pmatrix} 0 & 0.6 & 0.3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.4 & 0 & 0 \\ 0.7 & 1 & 0.9 & 0 \end{pmatrix} \end{array}$$

$Q(John, Mary) = 0.6$ means $John$ is considerably "much older than" $Mary$, and $Q(Thomas, John) = 0$ implies that $Thomas$ is not at all "much older than" $John$. The matrix for this relation is not symmetric at all. Such a nonsymmetric fuzzy relation is visualized by a directed graph with weights

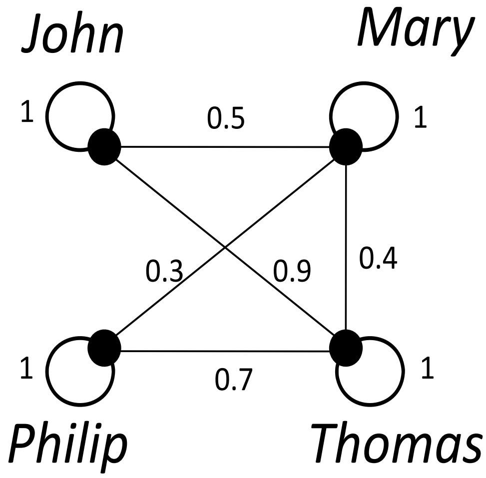

Figure 3.17: Undirected fuzzy graph which expresses the relation R on $X = \{J, M, T, P\}$

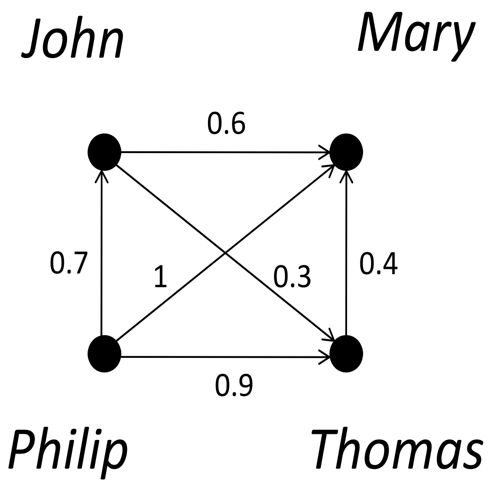

Figure 3.18: Fuzzy digraph which expresses the relation Q on $X = \{J, M, T, P\}$

as in Fig. 3.18, and is called a fuzzy digraph, although the name *fuzzy graph* in the broad sense refers to both fuzzy undirected graph and fuzzy digraph.

Generally, given a finite universal set $X = \{x_1, x_2, \ldots, x_n\}$ and a fuzzy relation R on X, the fuzzy graph is a digraph on X with weight $R(x_i, x_j)$ on the edge (x_i, x_j). If $R(x_i, x_j) = 0$, we do not draw the edge (x_i, x_j) in the figure. A fuzzy digraph may have two edges between a pair of vertices, and a loop on a vertex. Consider an example of $X = \{x_1, x_2\}$ and

$$R = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} 0.5 & 1 \\ 0.8 & 0 \end{pmatrix} \end{array}$$

which is visualized as Fig. 3.19.

Let X be finite or infinite for the moment. A fuzzy relation R is called reflexive if

$$R(x, x) = 1 \quad \forall x \in X$$

(Note that the symbol $\forall$ is read as *for all.*)

R is called symmetric if

$$R(x, y) = R(y, x) \quad \forall x \in X$$

A reflexive and symmetric fuzzy relation on a finite universe can be visualized, not by a fuzzy digraph, but by a fuzzy (undirected) graph.

Another type of a fuzzy graph is used for visualizing a fuzzy relation, say R, on $X \times Y$, where X and Y are considered to be two different universes.

Consider an example in which $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. The relation R on $X \times Y$ is given by

$$R = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} 1 & 0.3 \\ 0 & 0.4 \end{pmatrix} \end{array}$$

Then the corresponding fuzzy graph is shown in Fig. 3.20.

Since a fuzzy relation in a fuzzy set of the product space, all fuzzy set operations are applied to fuzzy relations. Furthermore, there are proper operations to fuzzy relations, the most important of which is the max-min composition. The max-min composition is frequently abbreviated as the composition. To discuss the composition, let us review briefly the composition of crisp relations. Let P and Q are crisp relations, for the moment, on

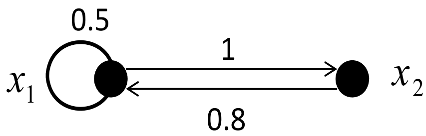

Figure 3.19: Fuzzy digraph R with a loop and two edges between x_1 and x_2

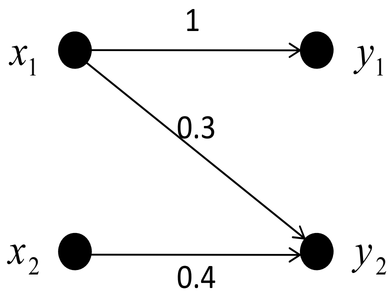

Figure 3.20: Fuzzy graph R on $X \times Y$

$X \times Y$ and $Y \times Z$, respectively. The composition of P and Q, denoted by $P \circ Q$, is defined as follows. For $x \in X$ and $z \in Z$, $P \circ Q(x,z) = 1$ if for some $y \in Y$, $P(x,y) = 1$ and $Q(y,z) = 1$; otherwise $P \circ Q(x,z) = 0$.

Let us turn to fuzzy case and assume now that P and Q are fuzzy relations on the same universes. In view of the crisp composition, it is natural to consider the fuzzy composition in terms of α-cuts. Thus, we should define $P \circ Q$ as:

"For any $\alpha \in [0,1]$, $(P \circ Q)_\alpha(x,z) = 1$ if $P_\alpha \circ Q_\alpha(x.z) = 1$; otherwise $(P \circ Q)_\alpha(x,z) = 0$."

Most of textbooks in fuzzy sets define the composition without the use of α-cuts. Namely, $P \circ Q$ is defined to be

$$(P \circ Q)(x,z) = \max_{y \in Y} \ \min\{P(x,y), Q(y,z)\}$$

Using $\vee$ and $\wedge$, the same equation is written as

$$(P \circ Q)(x,z) = \bigvee_{y \in Y} P(x,y) \wedge Q(y,z)$$

(Since this defnition uses max and min operations, the composition is called the max-min composition.) The latter definition without an α-cut is more elegant than the former with α-cuts; it is shown that these definitions are equivalent.

To see this, it is sufficient to note that $P \circ Q$ by the latter definition satisfies

$$(P \circ Q)_\alpha = P_\alpha \circ Q_\alpha$$

for any $\alpha \in [0,1]$.

The last equation is proved as follows. For any $\alpha \in [0,1]$,

$$\begin{aligned}
(P \circ Q)_\alpha(x,z) = 1 \quad &\Longleftrightarrow \quad \exists y \in Y, \min\{P(x,y), Q(y,z)\} \geq \alpha \\
&\Longleftrightarrow \quad \exists y \in Y, P(x,y) \geq \alpha \textit{ and } Q(y,z) \geq \alpha \\
&\Longleftrightarrow \quad \exists y \in Y, P_\alpha(x,y) = 1 \textit{ and } Q(y,z)_\alpha = 1 \\
&\Longleftrightarrow \quad \exists y \in Y, (P_\alpha \circ Q_\alpha)(x,z) = 1
\end{aligned}$$

Let P, Q, R be fuzzy relations on appropriately defined universes. Notice that for crisp relations $P_\alpha, Q_\alpha, R_\alpha$, the asscociative property is valid:

$$(P_\alpha \circ Q_\alpha) \circ R_\alpha = P_\alpha \circ (Q_\alpha \circ R_\alpha)$$

Now, for an arbitrary $\alpha \in [0,1]$,

$$\begin{aligned}[(P \circ Q) \circ R]_\alpha &= (P \circ Q)_\alpha \circ R_\alpha \\ &= (P_\alpha \circ Q_\alpha) \circ R_\alpha \\ &= P_\alpha \circ (Q_\alpha \circ R_\alpha) \\ &= P_\alpha \circ (Q \circ R)_\alpha \\ &= [P \circ (Q \circ R)]_\alpha\end{aligned}$$

Seeing that two fuzzy sets are equal if and only if any α-cut of them are equal, we have

$$(P \circ Q) \circ R = P \circ (Q \circ R)$$

Thus we have established the associativity of the composition. It should be noted that the above property is proved in the same way as the argument by which the distributivity of fuzzy sets has been proved.

In this way, α-cut is essential in consideration of fuzzy sets ans fuzzy relations. Consequently, the meaning of fuzzy graphs (and digraphs) should also be described in terms of *alpha*-cuts. Let us consider the example in Fig. 3.20 again. Unlike ordinary graphs with weights in which the weight means the amount of a flow, the grade in a fuzzy graph is degree of attainability. If we cut the fuzzy graph in Fig. 3.20 by various α, we have a collection of crisp graphs. Let $\alpha = 0.3$, then x_1 and y_2 are connected, but if $\alpha = 0.5$, then there is only one edge in the α-cut, that is, (x_1, y_1). The degree of attainability between x_i and x_j means the maximum value of α such that x_i and x_j are connected in that α-cut of the fuzzy graph.

Since the grade in a fuzzy graph is degree of attainability, the composition of two fuzzy graphs should be interpreted in the same manner. Indeed, the simplest way to see the meaning of the max-min composition is to observe fuzzy graphs. Let us consider an example of two relations P on $X \times Y$ and Q on $Y \times Z$, where $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, $Z = \{z_1, z_2\}$,

$$P = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} 0.5 & 0.7 \\ 0.3 & 1.0 \end{pmatrix} \end{array}$$

$$Q = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{pmatrix} 0.4 & 0.7 \\ 0.8 & 0.6 \end{pmatrix} \end{array}$$

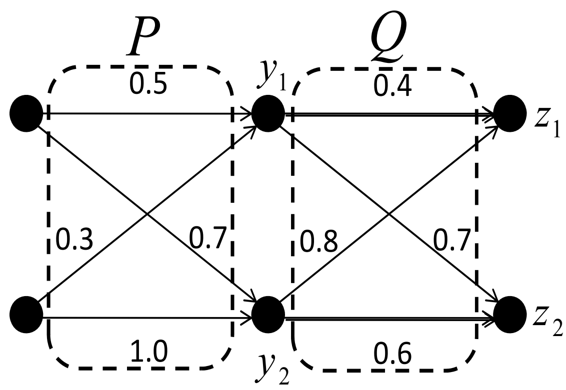

Figure 3.21: Fuzzy graphs P and Q that are combined

As in Fig. 3.21, the two fuzzy graphs P and Q are combined into one fuzzy graph.

By the analogy of the attainability in a fuzzy relation, the grade $(P \circ Q)(x_i, z_k)$ should imply the degree of attainability between x_i and z_k through y_j, $j = 1, 2$.

Take x_1 and z_1 to consider $(P \circ Q)(x_1, z_1)$. Remember that the degree of attainability means the maximum value of α such that x_1 and z_1 are connected in that α-cut. There are two paths $x_1 - y_1 - z_1$ and $x_1 - y_2 - z_1$. The attainability using $x_1 - y_1 - z_1$ is $\min\{R(x_1, y_1), Q(y_1, z_1)\} = \min\{0.5, 0.4\} = 0.4$, since if $\alpha > 0.4$, one of edges (x_1, y_1) and (y_1, z_1) disappears in that α-cut of the fuzzy graph. The degree using $x_1 - y_2 - z_1$ is $\min\{R(x_1, y_2), Q(y_2, z_1)\} = \min\{0.7, 0.8\} = 0.7$ by the same reason. Now, to reach from x_1 to z_1, we can use either of the two paths, and consequently the latter path $x_1 - y_2 - z_1$ is more favorable, since the latter path has the greater degree of attainability. Namely, the overall degree of attainability is

$$\begin{aligned} & \max\{\min(R(x_1, y_1), Q(y_1, z_1)), \min(R(x_1, y_2), Q(y_2, z_1))\} \\ = \; & \max\{0.4, 0.7\} = 0.7 \end{aligned}$$

The last equation is just the expression of the max-min composition defined before.

Generally, given $X = \{x_1, x_2, \ldots, x_\ell\}$, $Y = \{y_1, y_2, \ldots, y_m\}$, $Z = \{z_1, z_2, \ldots, z_n\}$, P on $X \times Y$, and Q on $Y \times Z$, we have m paths of $x_i - y_j - z_k$, $j = 1, 2, \ldots, m$, to connect x_i and z_k. The attainability through $x_i - y_j - z_k$ is $\min\{P(x_i, y_j), Q(y_j, z_k)\}$, and the overall attainability is

$$\max_{y_j \in Y} \; \min\{P(x_i, y_j), Q(y_j, z_k)\}$$

since we can use any of these m paths. Thus, $\min\{P(x_i, y_j), Q(y_j, z_k)\}$ describes attainability of a path, and $\max_{y_j \in Y}$ implies that we can choose the path of the maximum attainability out of possible routes.

Apart from the graphical interpretation, algebraic calculation of the composition should use $\wedge$ and $\vee$ instead of min and max. Thus, in hte former example,

$$P \circ Q = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} \quad z_1 \quad\quad\quad\quad\quad\quad\quad & \quad\quad\quad\quad z_2 \end{array} \\ \begin{pmatrix} (0.5 \wedge 0.4) \vee (0.7 \wedge 0.8) & (0.5 \wedge 0.7) \vee (0.7 \wedge 0.6) \\ (0.3 \wedge 0.4) \vee (1.0 \wedge 0.8) & (0.3 \wedge 0.7) \vee (1.0 \wedge 0.6) \end{pmatrix} \end{array} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{pmatrix} 0.7 & 0.6 \\ 0.8 & 0.6 \end{pmatrix} \end{array}$$

Sometimes $\wedge$ is replaced by a dot ($\cdot$) and $\vee$ is replaced by plus ($+$). Then the same calculation is written as

$$P \circ Q = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} \quad z_1 \quad & \quad\quad z_2 \end{array} \\ \begin{pmatrix} 0.5 \cdot 0.4 + 0.7 \cdot 0.8 & 0.5 \cdot 0.7 + 0.7 \cdot 0.6 \\ 0.3 \cdot 0.4 + 1.0 \cdot 0.8 & 0.3 \cdot 0.7 + 1.0 \cdot 0.6 \end{pmatrix} \end{array} = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{pmatrix} 0.7 & 0.6 \\ 0.8 & 0.6 \end{pmatrix} \end{array}$$

It is interesting to note that the way of using the symbols $+$ and $\cdot$ in the last calculation is the same as the ordinary matrix calculation, although the result is different from the ordinary matrix product.

Given a fuzzy set A of X and a fuzzy relation R on $X \times Y$, the max-min composition

$$(A \circ R)(y) = \max_{x \in X} \min\{A(x), R(x, y)\}$$

is used in fuzzy inference. Figure 3.22 is the fuzzy graph representing $A \circ R$ for

$$A = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{pmatrix} 0.4 & 0.5 \end{pmatrix} \end{array} \qquad R = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{pmatrix} 0.2 & 0.7 \\ 0.3 & 0.6 \end{pmatrix} \end{array}$$

The fuzzy set $A(x)$ is represented by weights on two edges from a dummy vertex λ to x_1 and x_2. The resulting membership values of $A \circ R$ is written in parentheses by the vertices y_1 and y_2.

Algebraically,

$$\begin{aligned} A \circ R &= \begin{pmatrix} 0.4 & 0.5 \end{pmatrix} \begin{pmatrix} 0.2 & 0.7 \\ 0.3 & 0.6 \end{pmatrix} \\ &= \begin{pmatrix} (0.4 \wedge 0.2) \vee (0.5 \wedge 0.3) & (0.4 \wedge 0.7) \vee (0.5 \wedge 0.6) \end{pmatrix} \\ &= \begin{pmatrix} 0.3 & 0.5 \end{pmatrix} \end{aligned}$$

If A is a fuzzy set on real numbers $X = \boldsymbol{R}$ and R is a fuzzy relation on $X \times Y = \boldsymbol{R}^2$, the composition is described by such a figure as Fig. 3.23. In this figure the triangle on the x-axis in the left side is the fuzzy set $A(x)$. The large tetrahedron is the relation R. Then the triangle with the shadow on Y-axis is $A \circ R$.

To obtain $A \circ R$, we consider the cylindrical extension

$$A'(x, y) = A(x) \quad for\ any \quad y \in Y$$

and take $A' \cap R$. Notice that

$$(A' \cap R)(x, y) = A'(x, y) \wedge R(x, y) = A(x) \wedge R(x, y).$$

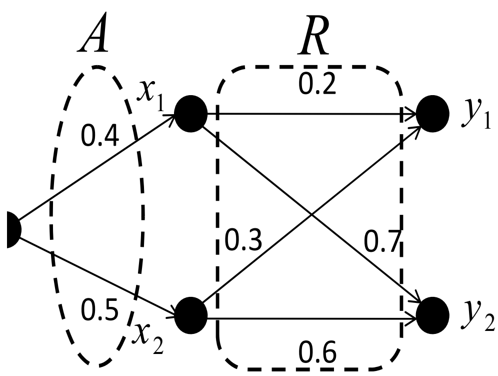

Figure 3.22: Fuzzy graph for calculating $A \circ R$, where A is a fuzzy set of X

In Fig. 3.23, $A' \cap R$ is a small tetrahedron, a part of R, which neighbors A and surrounded by solid and broken lines. Then the shadow projected onto the plane of $Y \times [0, 1]$ shows the membership function of $A \circ R$. Namely, the projection of $A' \cap R$ is given by

$$\max_{x \in X} (A' \cap R)(x, y) = \max_{x \in X} A(x) \wedge R(x, y)$$

which is just the max-min composition.

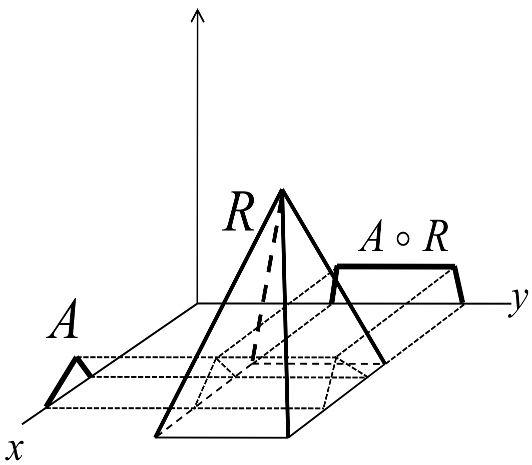

Figure 3.23: Geometrical calculation of $A \circ R$ when A is a fuzzy set of real numbers and R is a fuzzy relation on $\boldsymbol{R}^2$